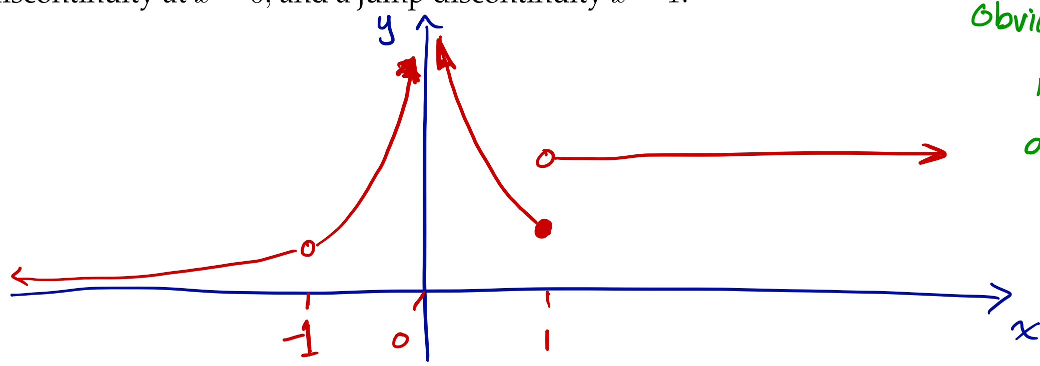
Lecture Notes 2-5: Continuity (Day 2)

REVIEW: A function $f(x)$ is continuous at the number $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Sketch a function with domain $(\infty, -1) \cup (-1, \infty)$ that has a removable discontinuity at $x = -1$, an infinite discontinuity at $x = 0$, and a jump discontinuity $x = 1$.

Obviously, there are many correct answers here.



GOALS: In this lesson, we will practice using the definition of continuity, define right- and left-continuity, and learn (& apply) several very powerful theorems concerning continuous functions.

DEFINITION: A function $f(x)$ is **continuous from the right** at the number $x = a$ if

$$\boxed{\lim_{x \to a^+} f(x) = f(a)},$$

A function $f(x)$ is **continuous from the left** at the number $x = a$ if

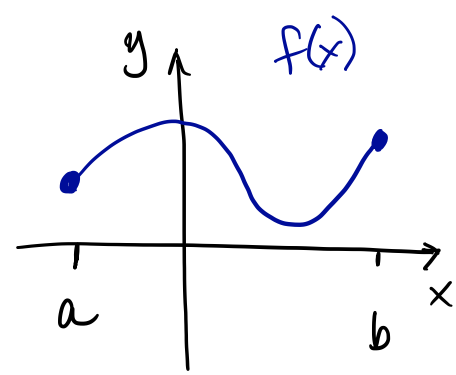
$$\boxed{\lim_{x \to a^-} f(x) = f(a)},$$

QUESTION: Look at your picture above and determine all the a-values for which your function is continuous from the right and those for which your function is continuous from the left.

In my picture, the function is continuous from the left at $x=1$.
Also, everywhere the function is continuous, the function is continuous from the right and the left.

QUESTION: Why would we want one-sided continuity?

So we can say a function is continuous on an interval. See example →
The function $f(x)$ is continuous on $[a,b]$.
(It's right continuous at $x=a$ and left continuous at $x=b$.)



QUESTION: Assume $f(x)$ and $g(x)$ are BOTH continuous at $x = a$, what do you think should be true about the new function $H(x) = f(x) + g(x)$ and how would you JUSTIFY your intuition?

It seems like $H(x)$ should also be continuous.

Justification:

$H(a) \overset{\text{definition of } H(x)}{=} f(a) + g(a)$ and

$$\lim_{x\to a} H(x) = \lim_{x\to a}\left[f(x)+g(x)\right] \overset{\text{rules about limits}}{=} \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\overset{\text{because } f \text{ and } g \text{ are continuous}}{=} f(a) + g(a)$$

THEOREMS 4, 5, AND 7 (as numbered in your textbook) all tell us that a large family of familiar functions are continuous. Below we will list this collection. The numbering aligns with the textbook theorem.

4.1 $f(x)+g(x)$

4.2 $f(x)-g(x)$

4.3 $c\,f(x)$

4.4 $f(x)\cdot g(x)$

4.5 $\dfrac{f(x)}{g(x)}$ provided $g(a)\neq 0$

5a All polynomials are continuous everywhere. $(-\infty,\infty)$

5b All rational functions are continuous where they are defined. (denominator $=0$ is a problem)

7 roots, trig functions, inverse trig functions, exponential and logarithmic functions continuous where defined.

PRACTICE PROBLEMS:

1. Determine the intervals over which the function $f(x) = \frac{3e^x+\tan x}{5x}$ is continuous and *justify* your answer using the Theorems above.

$\boxed{7}$ implies e^x and $\tan x$ are continuous provided $x \neq \frac{k\cdot\pi}{2}$, odd integer k. So $\boxed{4.3}$ and $\boxed{4.1}$ imply $3e^x+\tan x$ is continuous. Now $\boxed{4.5}$ says $\frac{3e^x+\tan x}{5x}$ is continuous provided $x\neq 0$.

Answer: $\ldots\left(-\frac{5\pi}{2},-\frac{3\pi}{2}\right)\cup\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)\cup\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},\frac{5\pi}{2}\right)\cup\ldots$

2. Evaluate $\displaystyle\lim_{x\to\pi/4}\frac{3e^x+\tan x}{5x}$ and *justify* your strategy.

$$\lim_{x\to\frac{\pi}{4}}\frac{3e^x+\tan x}{5x} = \frac{3e^{\pi/4}+\tan\left(\frac{\pi}{4}\right)}{5\pi/4} = \frac{4\left(3e^{\pi/4}+1\right)}{5\pi} = \boxed{\frac{4}{5\pi}\left(3e^{\pi/4}+1\right)}$$

Justification: The function is continuous at $x=\pi/4$, so we can just PLUG IN.

THEOREMS 8 AND 9 (as numbered in your textbook) tell us that continuity is preserved by function *composition* **provided** the resulting function is defined.

EXAMPLE: Determine all x-values for which the function $f(x) = \ln\left(\frac{1}{x} - 1\right)$ is continuous.

I will find x-values we must eliminate from the domain of f(x).
Since the domain of natural log is $(0,\infty)$, we require $\frac{1}{x} - 1 > 0$ or $\frac{1}{x} > 1$.

So $x < 1$ and $x > 0$.

Answer: The domain of f(x) is $(0,1)$. Since f(x) is the composition of continuous functions, f(x) is continuous on $(0,1)$.

PRACTICE PROBLEMS:

1. Determine the domain of the function $g(r) = \tan^{-1}(1 + e^{-r^2})$ and explain why $g(r)$ is continuous at every number in its domain.

thinking:
- arctan(x) has domain $(-\infty,\infty)$. So $1+e^{-r^2}$ can be anything.
- e^x has domain $(-\infty,\infty)$ so $-r^2$ can be anything.

domain of g(r): $(-\infty,\infty)$.
g(r) is continuous for all real numbers because
$y = \tan^{-1}x$ and $y = e^x$
and $y = 1+x$ are all continuous and g(r) is the composition of these.

2. Use continuity to evaluate $\lim_{x\to 4} 3^{\sqrt{x^2-2x-4}}$.

Using continuity, we can plug in.

So $\lim_{x\to 4} 3^{\sqrt{x^2-2x-4}} = 3^{\sqrt{16-8-4}} = 3^2 = \boxed{9}$

3. Let $f(x) = 1/x$ and $g(x) = 1/x^2$. (a) Find $(f \circ g)(x)$. (b) Explain why $f \circ g$ is not continuous everywhere.

(a) $(f\circ g)(x) = f\left(\frac{1}{x^2}\right) = \dfrac{1}{\frac{1}{x^2}} = x^2$

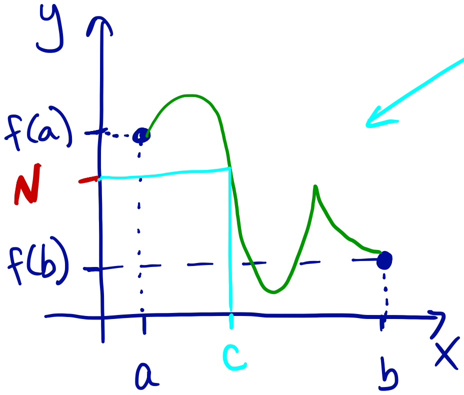
(b) g(x) is not defined at $x=0$.
So $(f\circ g)(x)$ is continuous on $(-\infty,0)\cup(0,\infty)$

$\boxed{\text{THE INTERMEDIATE VALUE THEOREM:}}$ Suppose $f(x)$ is a function such that

- $f(x)$ is continuous on $[a, b]$,
- $f(a) \neq f(b)$, and
- N is a number between $f(a)$ and $f(b)$,

then,

picture version:



conclusion

there exists some number c in the open interval (a,b) so that $f(c) = N$.

Note: to use this theorem you have to show (make sure) all three bullet points above hold.

$\boxed{\text{PRACTICE PROBLEMS:}}$

1. Use the Intermediate Value Theorem to show that the equation $x^4 + x - 3 = 0$ must have a root in the interval $(1, 2)$.

a=1 and b=2

$f(x) = x^4 + x - 3$

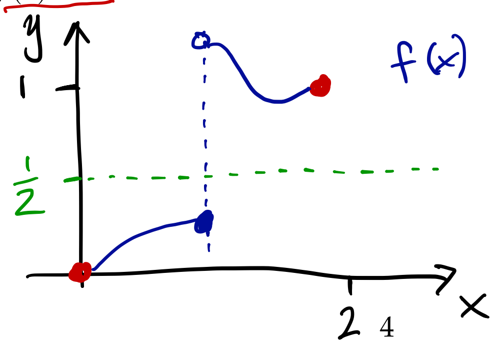
- $f(x) = x^4 + x - 3$ is a poly & thus continuous everywhere
- $f(1) = 1 + 1 - 3 = -1$
  $f(2) = 16 + 2 - 3 = 15$
  So $f(1) \neq f(2)$
- $N = 0$ and $-1 = f(a) < 0 < f(b) = 15$

(All three bullet points hold.)

Conclusion:

There is some c in $(1,2)$ so that $f(c) = 0$. Thus, the equation has a solution in the interval $(1,2)$.

2. Give an example of a function $f(x)$ that is defined for every number in the interval $[0, 2]$ such that $f(0) = 0$, $f(2) = 1$ but there does not exist a single x-value in the interval $(0, 2)$ such that $f(x) = 1/2$.



Note: $f(x)$ is not continuous!